Concave majorant of stochastic processes and Burgers turbulence

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Abstract
In this paper, we study the convex hull of the graph of a stochastic process, and more precisely its extremal points. The times where those extremal points are reached, called extremal times, form a negligible set for Lévy processes, their integrated processes, and Itô processes. We examine more closely the case of a Lévy process with bounded variation. Its extremal superior points are almost surely countable, with accumulation only around the extremal values.

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If a smooth drift f is added to the process, the extremal points are characterised by an equality between the derivative of the concave majorant and that of the drift, f', generalising the no drift case result.

These results are related to the study of a fluid ruled by Burgers equation in vanishing viscosity.

Keywords: Lévy processes, bounded variation, concave majorant, extremal points, convex hull, Burgers equation.

AMS: 60G17, 60D05, 35Q35, 35R60, 60J99.

Introduction and notation

In $\mathbb{R}^d$, $B(0,\epsilon)$ denotes the open ball with centre 0 and radius $\epsilon > 0$, $\mathrm{int}(A)$ is the interior of a set A, $\mathrm{cl}(A)$ is its closure and ∂A is its border. For a function f, $\mathrm{gr}(f)$ denotes its graph. If a function g admits left and right limits in a point a, denote by $g(a^-)$ (resp. $g(a^+)$) its limit on the left (resp. right) of a.

There are multiple reasons to study the convex hull of a random process X. From a geometric point of view, it is a good indicator of the graph structure, both locally and globally, and captures some features of the regularity of X. Groenboom [6] has first studied this topic if X is a standard brownian motion and $I = \mathbb{R}^+$, establishing the essential results, and Pitman [8] gave a simple and exhaustive description of the concave majorant. Davydov [5] studied these matters if X is a Gaussian process.

In fluid mechanics, this geometrical problem is closely related to the solution of the one-dimensional Burgers equation with vanishing viscosity. Namely, if $v(x,t)$ is the velocity field of an incompressible fluid, Burgers has introduced the equation

$$\partial_t v + \partial_x \left(\frac{v^2}{2}\right) = \epsilon \partial_{xx}^2 v \tag{1}$$

as a simple model of hydrodynamic turbulence, where ϵ is the viscosity. Burgers equation is a simplified version of Navier-Stokes equation, and even if it is known among physicists that it does not describe turbulence very accurately, it is broadly used in many physical problems such as shock wave formation in compressible fluids, or formation of large scale structure of the universe. See [13] for more detailed information. Frisch explained why a probabilistic description of turbulence is necessary, and there has been since the 80's an abundant literature about the solution of Burgers equation with random initial conditions and vanishing viscosity, defined as the limit of the solution as ϵ tends to 0.

It is explained in section 2 how a topological description of the fluid can be deduced from the study of the convex hull of $\psi + f$, where $\partial_a \psi(0,.) = -v$, and f is a parabolic drift. If the drift is removed, the study of the convex hull of ψ retrieves the asymptotic structure of the fluid when time goes to ∞.

For a function f on an interval of $\mathbb{R}$, call $\mathcal{E}_f$ the set of extremal points of the convex hull C_f of f's graph . The set $\mathcal{E}_f$ can be decomposed in $\mathcal{E}_f^+ \cup \mathcal{E}_f^-$, where $\mathcal{E}_f^+$ (resp. $\mathcal{E}_f^-$) lies in the upper (resp. lower) border of C_f. Denote by $\overline{f}$ the smallest concave function greater than f, or *concave majorant*, and $\underline{f} = -(\overline{-f})$ its convex minorant. We also have $\mathcal{E}_{\overline{f}} = \mathcal{E}_f^+$ and $\mathcal{E}_{\underline{f}} = \mathcal{E}_f^-$. Call $\mathbf{E}_f$ (resp. $\mathbf{E}_f^+, \mathbf{E}_f^-$) the projection of $\mathcal{E}_f$ (resp. $\mathcal{E}_f^+, \mathcal{E}_f^-$) onto the horizontal axis.

It is shown in this paper that, if X belongs to a certain class of Markov processes, called *reversible*, then $\mathbf{E}_X$ is a.s negligible, and so it is for its integrated process. The same result holds if X is an Itô process. The main result of this paper concerns the case where X is a Lévy process with bounded variation. The set $\mathbf{E}_X^+$ is a.s countable with accumulation only possible at the time T where X tends to its extremal values. We also give sufficient and necessary conditions for T to be an accumulation point of $\mathbf{E}_X^+$ on its right or on its left. More generally, if a sufficiently smooth deterministic function f is added up to X, we show that the only right-(resp. left-)accumulation points of $\mathbf{E}_X^+$ can occur only at times a where $f'(a) = (\overline{X+f})'(a^+)$ (resp. $f'(a) = (\overline{X+f})'(a^-)$).

In the preliminaries, we introduce rigourously all the notation used in the paper, and recall some facts about Lévy processes which will be useful for the sequel. Then, in section 2, independant of the rest of the paper, we explain the relation between the convex hull of a function and the description of the shock structure of a fluid without viscosity. Although it is not an accurate model, the discrete case is presented as well to help capture the main features of the problem. Finally, we state the results concerning the convex hull for some random processes, with a complete description for a Lévy process with bounded variation.

In this paper, we investigate the metrical and topological properties of the set of extremal times $\mathbf{E}_X$ of a càdlàg random process $\{X(a)\ ;\ a \in I\}$, for an interval I of $\mathbb{R}$. The symmetry of the problem allows us to simply study $\mathbf{E}_X^+$, the properties of $\mathbf{E}_X^-$ being similar. In this respect, define $X^*(a) = \max(X(a^-), X(a))$, so that $a \in \mathbf{E}_X^+$ iff $(a, X^*(a))$ is an extremal point of the graph of X.

For a process X defined on a right neighbourhoud of a point $a \in \mathbb{R}$, eventually random, denote by $R_a^+(X)$ the event

$$\exists \epsilon > 0,\ X(s) > X(a) \text{ for } a < s < a + \epsilon,$$

and $R_a^-(X) = R_a^+(-X)$. If the process satisfies Blumenthal's *zero-one law*, which is the case for "nice" Markov processes, then they both are trivial. Furthermore, for Lévy processes, the probability of these events do not depend on

a if it is a deterministic time, or more generally a stopping time, in this case simply write $R^+(X)$ and $R^-(X)$.

In all the article, for a process X defined on an interval I, the notation M_a stands for $(a, X^*(a))$, where a belongs to I and might be random.

1 Preliminaries

1.1 Lévy processes

The results obtained for Lévy processes distinguish the cases where the paths have bounded or unbounded variation.

For a function f on an interval I, f is said to have bounded variation if for any bounded subinterval J there is a finite constant m_J such that for any sequence $a_1 < ... < a_q$ in J,

$$\sum_{i=1}^{q-1} |f(a_{i+1}) - f(a_i)| \leq m_J.$$

It is known (see [3]) that each Lévy process Y with bounded variation can be decomposed in the sum of two simple processes. We have

$$Y(a) = ba + X(a),\ a \in I$$

where the real number b is the *drift*, and X is the Poisson component, a pure jump Lévy process which characteristic exponent is of the form

$$\psi(\theta) = \int_{\mathbb{R}} (\exp(i\theta x) - 1)\nu(dx),$$

where ν is a measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} (1 \wedge |x|)\nu(dx) < \infty.$$

The measure ν is the *Lévy measure* of X and determines the size and frequency of the jumps of X.

The drift b is irrelevant in our study, because the set $\mathbf{E}_Y$ is does not depend of b, thus we will assume henceforth that $b = 0$, i.e $Y = X$ has no drift and only varies by jumps.

Such processes also enjoy the property of being "flat" around the origin (see for example [12]).

Lemma 1.1. *Let X be a Lévy process with bounded variation and no drift on an interval I. Then we have, for any $a \in I$, with probability* 1,

$$\lim_{x \to a} \frac{X(x) - X(a)}{x - a} = 0.$$

Furthermore, due to the strong Markov property,the previous relation still holds when a is replaced by a stopping time.

There is another relevant feature regarding the extremal set of Lévy processes.

Definition 1.1. *Let X be a Lévy process with bounded variation and no drift on an interval I, and ν its Lévy measure. The measure ν is said to be* downards dissymetric *if the event $R^+(X)$ occurs with probability* 1*. In the opposite case, if the Lévy measure of $-X$ is downards dissymetric, ν is* upwards dissymetric*. Otherwise, ν is* non-dissymetric*.*

This property is also known as the *irregularity of the half-line* for X. Bertoin [2] established an explicit characterisation of downwards dissymetric Lévy measures.

2 Fluid in vanishing viscosity

The study of the convex hull of a function f is connected to the shock structure of a fluid which initial potential is f. It is easier to understand the technical details in a discrete framework, i.e with discrete particles, but the continuous case is more realistic, and is studied in a second time.

2.1 Discrete case

Consider $\{P_k\ ;\ k \in \mathbb{Z}\}$, a set of particles disposed on the real line having respective initial positions $\{x_k\ ;\ k \in \mathbb{Z}\}$, with $x_k < x_{k'}$ for $k < k'$ and $|x_k| \to \infty$ when $|k| \to \infty$. Particle P_k has initial random velocity $v_k \in \mathbb{R}$ and mass 1. The rule of evolution is the following: Any clump of particles with velocity v and mass m evolves freely on the line until meeting another clump of velocity v' and mass m', in which case they form a new clump with mass $m + m'$ and velocity $\frac{mv+m'v'}{m+m'}$, so that mass and momentum are preserved.

Define the initial potential by $\psi(0) = 0$ and $\psi(k) - \psi(k-1) = -v_k$. We suppose that the laws of the random variables v_k yield that, for all $k < p < r$ in $\mathbb{Z}$, $\frac{\psi(p)-\psi(k)}{p-k} \neq \frac{\psi(r)-\psi(p)}{r-p}$ almost surely. It is the case for instance if any finite dimensional distribution $(v_{k_1}, .., v_{k_q})$ on $\mathbb{R}^q$, $q \geq 1$, charges no atom. We still call ψ the linear interpolation of ψ on $\mathbb{R}$, obtained by connecting the dots with segments.

Theorem 2.1. *Call $\mathbf{E}^+_\psi$ the set of extremal superior times of ψ. There is a partition of the set of integers $\mathbb{Z} = \cup_{x \in \mathbf{E}^+_\psi} Z_x$, where Z_{x_q} is the set of indices of all particles that end up in the same clump as P_q after a finite time. Note that q is the left extremity of Z_{x_q}. In particular, a particle P_k never collides with a neighbour iff x_k and x_{k+1} are in $\mathbf{E}^+_\psi$.*

This theorems enlightens the fact that the study of the concave majorant of ψ brings all the information needed for an asymptotic topological description of the fluid.

Proof. To understand the behaviour of this particle system when time goes to ∞, we investigate conditions on v_k so that particle P_k hits its left neighbour P_{k-1}, for some k in $\mathbb{Z}$. The key remark is that a clump cannot be slowed down by a collision coming from the left, because if it happens, it means that the other clump involved had a higher speed. It is simple to see that there will be a collision between P_k and P_{k-1} if $v_{k-1} > v_k$. But it is not the only case where it will occur. What happens if for example P_k has a sufficiently high initial speed to avoid P_{k-1}, but hits P_{k+1}, and hence is slowed down? It will hit P_{k-1} if $v_{k-1} > \frac{v_k+v_{k+1}}{2}$. Also, P_{k-1} might be helped by P_{k-2} to catch up P_k, and the corresponding new conditions are $\frac{v_{k-2}+v_{k-1}}{2} > v_k$ and $\frac{v_{k-2}+v_{k-1}}{2} > \frac{v_k+v_{k+1}}{2}$. Actually, the necessary and sufficient conditions for P_k and its clump to infinitely avoid a collision with P_{k-1}, and maybe other particles $P_j, j < k$, is

$$\frac{\psi(k-1) - \psi(k-1-u)}{u} > \frac{\psi(k+s) - \psi(k)}{s}, \; s, u \in \mathbb{N}^*$$

so that the clump involving P_{k-1} stays away from the one containing P_k. The equality cases almost never happen, due to the assumptions on the laws of the v_k.

Mathematically, this condition exactly means that k is an extremal superior time of ψ. Considering that P_k hits its right neighbour iff P_{k+1} hits its left neighbour, the study of $\mathbf{E}_\psi^+$ yields the full description of the shock structure of the system, after sufficiently large times.

□

2.2 Continuous case

Now, we assume that a fluid with viscosity ϵ is uniformly disposed on an interval I of the real line, and consider that it is ruled by Burgers equation (1). Here, $v(a,t)$ is the velocity field of the fluid at time t and point a. Hopf [7] and Cole [4] derived an explicit solution, stated in terms of the potential ψ defined by $\partial_a \psi(a,t) = -u(a,t)$ and $\psi(0,0) = 0$. The limit of this solution when the viscosity ϵ tends to 0 is given, for $x \in \mathbb{R}, t \geq 0$, by

$$\psi(x,t) = \sup_{a\in\mathbb{R}} \left[\psi(a,0) - \frac{(x-a)^2}{2t} \right] = -\frac{x^2}{2t} + \sup_{a\in\mathbb{R}} \left[\psi(a,0) - \frac{a^2}{2t} + \frac{xa}{t} \right]. \qquad (2)$$

Remark the similarity with the Legendre transformation of the function

$$\psi_t(a) = \psi(a,0) - \frac{a^2}{2t}.$$

The physical interpretation is similar to the discrete case, see [1] for details and proofs. For x real, let $a(x,t)$ be the supremum of points where the maximum in (2) is achieved, and $x(a,t)$ its right-continuous inverse. To help comprehend the situation, note that when $\overline{\psi}_t$ is a smooth strictly concave function, $x(a,t) = -\partial_a \overline{\psi}(a^+,t)$. The application $a \to x(a,t)$ is called Lagrangian, and $x \to a(x,t)$

inverse Lagrangian function. From the hydrodynamic point of view, $x(a,t)$ is the position at time t of a particle initially located at a. If a discontinuity of the inverse Lagrangian occurs at a point x: $a(x^-,t) < a(x,t)$, it means that all the particles initially located on $[a(x^-,t), a(x,t)[$ have formed a clump at point x at time t, and that is why such an interval is called a *shock interval*, and its extremities *shock points*. The left extremity a of such an interval satisfies $\overline{\psi}_t'(a-h) > \overline{\psi}_t'(a+h)$ for sufficiently small h, which exactly means that a is an extremal superior time of ψ_t. Conversely, this strict inequality indeed characterises left extremities of shock intervals.

Hence the set $\mathbf{E}^+_{\psi_t}$ contains the whole description of the shock structure of the fluid at time t, its points represent extremities of shock intervals if isolated on the right or left, while, on the contrary, the initial position of a particle not involved in any shock up to time t is isolated neither on its left nor on its right in $\mathbf{E}^+_{\psi_t}$. Notice the similarity of this result with theorem 2.1.

In conclusion, in order to obtain a complete description of the fluid at time t, our aim is to study $\mathbf{E}^+_{\psi+f}$, if ψ is the realisation of a Lévy process and f is the smooth drift $f(a) = -\frac{a^2}{2t}$ in the case of Burgers equation. What happens when time goes to ∞? The following result yields a form of continuity for $t \to \mathbf{E}^+_{\psi_t}$.

Proposition 2.1. *Let g be a function on a bounded interval I of $\mathbb{R}$, and f a continuous function on I. For a real number u, call $g_u = g + uf$ and $\mathbf{E}^+_u = \mathbf{E}^+_{g_u}$. Assume that*

$$\mathrm{cl}(\mathrm{gr}(g_u)) \cap \partial\mathcal{C}_{g_u} = \mathcal{E}_{g_u}, \tag{3}$$

i.e the graph of g_u only approaches the border of its convex hull in its extremal points. Then,

$$d_{\mathcal{H}}(\mathbf{E}^+_{u+h}, \mathbf{E}^+_u) \xrightarrow[h\to 0]{} 0,$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance between closed sets, defined by

$$d_{\mathcal{H}}(L,M) = \inf\{r>0\ ;\ L \subset M + B(0,r), M \subset L + B(0,r)\}.$$

Proof. Let $\epsilon > 0$. We need to find $\eta > 0$ such that $\mathbf{E}^+_{u-h} \subset \mathbf{E}^+_u + B(0,\epsilon)$ for $0 < h < \eta$. Since we can replace g by $-g$, it will be sufficient.

Since $\mathbf{E}^+_u$ is closed, $I \setminus \mathbf{E}^+_u$ can be decomposed into a countable set of disjoint open intervals $(I_{n,u})_{n\in\mathbb{N}}$. Put $K_{u,\epsilon} = I \setminus (\mathbf{E}^+_u + B(0,\epsilon))$. It intersects a finite number of $I_{n,u}$, the ones which size is greater than ϵ. Call $J_1, ..., J_q$ the closed intervals such that $K_{u,\epsilon} = \cup_{i=1}^q J_i$.

Take i in $\{1,...,q\}$. Since by hypothesis the closure of the graph of g_u does not touch that of $\overline{g_u}$ outside an extremal point, the graphs of g_u and $\overline{g_u}$ have positive distance above each $J_i, 1 \leq i \leq q$. In consequence, there is $h_i > 0$ such that the graphs of g_{u-h_i} and $\overline{g_{u-h_i}}$ are disjoints above J_i. In particular, $\mathbf{E}^+_{u-h_i} \cap J_i = \emptyset$. Taking $h = \min_i(h_i) > 0$ yields the result. $\square$

For typical realisations of the irregular random processes studied in this paper, (3) will be satisfied almost surely for almost all positive u. But also, with probability one, there will be some (random) u such that it is not satisfied. Hence, the previous theorem is only useful for us if we study the fluid for a fixed u. It is indeed the case, because the convex hull of $\psi(.,0)$ without drift corresponds to the case $u = 0$, or $t = \infty$. It is in fact the asymptotic case, and the previous theorem allows us to state that the structure of the fluid at sufficiently large times tends to the extremal set of $\psi(.,0)$, without drift. This is the purpose of the following corollary

Corollary 2.1. *When* $t \to \infty, \mathbf{E}^+_{\psi_t} \overset{d_{\mathcal{H}}}{\to} \mathbf{E}^+_{\psi}$.

The set $\mathbf{E}^+_{\psi}$ is studied in detail in the next section.

3 Extremal set of random processes

In a first time we show that the negligibility of the extremal set of Lévy processes and Itô processes occurs almost surely. Then the topology of the extremal set of Lévy processes with bounded variation is more profoundly investigated, and stronger results are obtained.

3.1 Negligibility results

To establish the negligibility, the main tool is Fubini's theorem, which enables us to use the following lemma.

In all this section, I stands for an interval of $\mathbb{R}$.

Lemma 3.1. *Let X be a separable process on I. If*

$$\forall a \in I, \quad \mathbb{P}(a \in \mathbf{E}^+_X) = 0,$$

then we have a.s $\lambda(\mathbf{E}^+_X) = 0$.

Proof. The application $(\omega, a) \to 1_{(a \,\in\, \mathbf{E}^+_X)}$ is measurable:

$$\left\{(\omega,a) \; ; \; a \notin \mathbf{E}^+_X(\omega)\right\} = \\ \left\{ (\omega, a) \; ; \; X(\omega,a) \leq \limsup_{\substack{s,v \;\in \mathbb{Q}^2 \\ s<a<v}} (a-s)X(\omega,v) + (v-a)X(\omega,s) \right\}.$$

Fubini's theorem gives us

$$0 = \int_I \mathbb{P}(a \in \mathbf{E}^+_X)\; dt = \int_\Omega \lambda(\{a \; ; \; a \in \mathbf{E}^+_X\})\; \mathbb{P}(d\omega) \tag{4}$$

and the proof is complete. ∎

Due to this lemma and the nice homogeneity properties of Lévy processes, we can establish the negligibility of the extremal set for any Lévy process, but the space homogeneity is actually not fully required, and the result holds under weaker hypotheses. Besides the strong Markov property, we require from a process X the triviality of the algebra $\bigcap_{s\downarrow 0}\sigma(X_s)$, which is called Blumenthal zero-one law. *We only consider Markov processes satisfying Blumenthal's zero-one law. It is also required that, a.e, the process is a.s continuous .*

Definition 3.1. *Let X be a Markov process. X is said to be **reversible** if, for any $a \in I$, the processes $\widehat{X}(s) = X(a+s) - X(a)$ and $\widecheck{X}(s) = X(a) - X((a-s)^-)$ (for s such that both expressions make sense) are Markov, and have the same distribution.*

For example, Lévy processes are reversible.

Theorem 3.1. *Let X be a reversible Markov process. Then*

- **(i)** $\lambda(\mathbf{E}_X) = 0$ *a.s.*
- **(ii)** *Let Z be a primitive of X. We have a.s $\lambda(\mathbf{E}_Z) = 0$.*

This result is proved in paragraph 4.1. Briefly, the idea is to use lemma 3.1 and so we have to prove that, any given a is almost never in the extremal set. The property of being in the extremal set depends geometrically on the past and the future of a, and the reversibility property ensures us that the conjoint behaviour of past and future is symmetric. The zero-one law for the process $\widehat{X} - \widecheck{X}$ finishes the proof.

The following result concerns Itô processes. We consider hereafter $I = \mathbb{R}^+$ for the sake of simpler statements.

Let $(B_a;\ a \geq 0)$ be a standard brownian motion and $\mathcal{F} = (\mathcal{F}_a;\ a \geq 0)$ its natural filtration.

Definition 3.2. *Let $\mathcal{M}_2^{loc}$ be the class of $\mathcal{F}$-adapted process ψ on $\mathbb{R}^+$ that satisfy*

$$\forall a > 0,\ \mathbb{E}\left(\int_0^a \psi^2(s)ds\right) < \infty.$$

Theorem 3.2. *Let X be an Itô process. X can be written under the general form*

$$X(a) = \int_0^a \psi(s)dB(s) + \int_0^a \phi(s)ds, \tag{5}$$

where $\psi, \phi \in \mathcal{M}_2^{loc}$.

Denote by $Y(a)$ the first term in (5) and $Z(a)$ the second one. Assume that, with probability one, there is no non-empty open interval where Y vanishes. Then, a.s $\lambda(\mathbf{E}_X) = 0$.

The hypothesis on Y is necessary because if $Y = 0$, Z can be chosen to be any smooth function, bringing a heavy extremal set. In other words, the irregularity of Y is sufficiently strong to overbalance the regularity of Z. In the proof (paragraph 4.2), we use lemma 3.1 again, showing that for any a, Y is not locally lipschitzian in a while Z is, so that the large fluctuations of Y around a make it a.s impossible for a to be in the extremal set.

3.2 Lévy processes with bounded variation

Lévy processes are often considered as initial data for Burgers turbulence. We focus here on the class of Lévy processes with bounded variation for technical reasons. Since it is irrelevant in our study, we also assume that the drift is null. In this case lemma 1.1 indicates the local behaviour of the process after zero, and consequently after each stopping time. Since the process is pure-jump, and its jumps are all simultaneously stopping times, it is easier to apprehend the structure of the graph with Markovian techniques.

Notice first that if the Lévy measure is finite, X is a.s piece-wise constant, and $\mathbf{E}_X$ is discrete in $\mathbb{R}^+$. For the sake of more simple statements, we assume in the sequel that the Lévy measure is infinite.

In this section, X is a Lévy process with bounded variation and no drift, and f is a smooth function. Hypotheses on f will be made more precise later. Let us start with the case where f is convex.

Lemma 3.2. *Let g and f be two functions on I, f being convex. Let a be an extremal superior time of $g + f$, then a is an extremal superior time of g.*

Proof. Let us pick s that is not extremal superior for g. We will show that s is not extremal superior for $g + f$ either. We can find $u < s < v$ and $\alpha, \beta = 1 - \alpha$ in $]0, 1[$ such that

$$s = \alpha u + \beta v,$$
$$g(s) < \alpha g(u) + \beta g(v).$$

The convexity of f gives us:

$$f(s) \leq \alpha f(u) + \beta f(v),$$

and by adding up we have

$$(g + f)(s) < \alpha(g + f)(u) + \beta(g + f)(v),$$

which proves that s is not extremal superior for $g + f$. □

Corollary 3.1. *If f is convex, $\mathbf{E}^+_{X+f} \subset \mathbf{E}^+_X \subset \mathbf{E}^+_{X-f}$.*

Adding up a convex function to X thins its extremal superior set, although adding a concave function widens it.

The next theorem sums up the main result obtained about concave functions:

Theorem 3.3. *Suppose that f is a concave function of class $\mathcal{C}^1$. Put $Y = X + f$.*

Then, for all $a \in \mathbf{E}_Y^+$, a is left isolated (resp. right isolated) in $\mathbf{E}_Y^+$ if $\overline{Y}'(a^-) \neq f'(a)$ (resp. $\overline{Y}'(a) \neq f'(a)$).

The proof is at section 4.3. The eventual accumulation points of $\mathbf{E}_Y^+$ are the points where the derivatives of $\overline{Y}$ and f exactly coincide.

With the help of corollary 3.1, this theorem can easily be generalised to the class of functions for which the interval I can be decomposed in a countable set of intervals above which f is either concave or convex.

Applying theorem 3.3 with $f = 0$ yields the asymptotic evolution of a fluid, as explained in the previous section. In this case, the only points where $\overline{Y}'(a) = f'(a)$ is the supremum of X. It is possible in this case to precise the topology of the convex hull around the maximal value, it depends on the the *regularity of the half-line* for X, see section 1.1. The following theorem gives an exhaustive topological description of the extremal set.

Theorem 3.4. *Let X be a bounded variation Lévy process with infinite Lévy measure and no drift on an interval I of $\mathbb{R}$. Let $T = \sup\{a \in I \mid X^*(a) = \sup_{u \in I} X(u)\}$. Then a.s $\mathbf{E}_X^+$ contains only T and jumping times of X, and its unique accumulation point is T.*

The time T is right isolated in $\mathbf{E}_X^+$ iff $\mathbb{P}(R^-(X)) = 1$, i.e if ν is downwards dissymmetrical, and left isolated iff $\mathbb{P}(R^+(X)) = 1$.

This theorem gives us a fairly good understanding of the shape of $X's$ concave majorant and extremal set. We do not have a complete quantitative description as [6] and [8] got for brownian motion, but we can fully understand the topological structure of the set.

physical interpretation: We consider here a fluid on an interval I, ruled by Burgers equation. The considerations of section 2.2 along with the results obtained in this section yield the following. Call $\mathbf{E}_{\psi_t}^+$ the set of shock points at time t. Remember that a shock point is the initial position of a particle which forms at time t the left extremity of a clump. Then $\mathbf{E}_{\psi_t}^+$ is decreasing with time (lemma 3.2). Let us assume now that the initial potential is a Lévy process with bounded variation. According to theorem 3.3 a shock point a at time t is isolated iff

$$-\frac{a}{t} \notin \{\overline{\psi}_t'(a^-), \overline{\psi}_t'(a)\}.$$

Also, the set $\mathbf{E}_{\psi_t}^+$ converges for the Hausdorff distance to $\mathbf{E}_{\psi}^+$ which is fully described by theorem 3.4. In particular, $\mathbf{E}_{\psi_t}^+$ tends to gather around a locally finite set of cluster points.

4 Proofs

We introduce the temporal translation operator θ:

Definition 4.1. *Let $T \in \mathbb{R}^+$, and X be a function on an interval I. Then we set*

$$\theta_T(X)(a) = X(T+a)$$

for a and T such thath $T + a \in I$.

4.1 Reversible Markov processes

Proof of theorem 3.1. Let $a \ \in I$. We have to show that a is almost never an extremal superior time.

We set $M_a = (a, X^*(a))$. By hypothesis, for a in a set of complete measure, with probability one, if a is fixed, $X^*(a) = a$ (i.e X is continuous in a), and we only consider ω for which $X^*(a) = a$ in the sequel.

We know that a is not an extremal superior time as soon as we can find $s > 0$ such that

$$X(a-s) + X(a+s) \geq 2X^*(a), \tag{6}$$

because then M_a would be under the segment $[M_{a-s}, M_{a+s}]$. We will show that we can a.s find s such that (6) is satisfied.

We set, for $s \geq 0$,

$$\widehat{X}(s) = X(a+s) - X(a), \ \check{X}(s) = X(a) - X((a-s)^-)$$

and

$$Y(s) = \widehat{X}(s) - \check{X}(s) = X((a-s)^-) + X(a+s) - 2X(a).$$

By reversibility of X, $\check{X}$ and $\widehat{X}$ have the same law and are Markov processes for s small enough.

By the strong Markov property, for $x \in \mathbb{R}$, the conditional processes $\widehat{X}_x = (\widehat{X}|X(a) = x)$ and $\check{X}_x = (\check{X}|X(a) = x)$ are independent. As a consequence, $Y_x = \widehat{X}_x - \check{X}_x$ is Markov and symmetric ($-Y_x \overset{(d)}{=} Y_x$). Then we have $\mathbb{P}(R^+(Y_x)) = \mathbb{P}(R^-(Y_x))$. These two events are trivials. Since thay are disjoint, they both have probability 0.

We have $\mathbb{P}(R^-(Y)) = \int_{\mathbb{R}} \mathbb{P}(R^-(Y_x))\mathbb{P}(X(a) \in \mathrm{d}x) = 0$ and so we can a.s. find s arbitrarily close to 0 such that $Y(s) \geq 0$, which proves that X satisfies (6). So, a.s $a \notin \mathbf{E}_X^+$. Thanks to lemma 3.1 we can conclude that a.s $\lambda(\mathbf{E}_X^+) = 0$.

Given that $-X$ satisfies the same hypotheses, a.s $\lambda(\mathbf{E}_{-X}^+) = \lambda(\mathbf{E}_X^-) = 0$ and a.s $\lambda(\mathbf{E}_X) = 0$, which proves (i).

Let us now consider $Z(a) = \int_0^a X(u)\mathrm{d}u$. To prove (ii), we have to show, as previously, that

$$a.s, \ \exists s > 0, \ \int_{a-s}^{a} X(u)\mathrm{d}u \geq \int_a^{a+s} X(u)\mathrm{d}u.$$

We have

$$\int_a^{a+s} X(u)\mathrm{d}u = \int_0^s \widehat{X}(u)\mathrm{d}u = \widehat{Z}(s) \quad \text{and} \quad \int_{a-s}^a X(u)\mathrm{d}u = \int_0^s \check{X}(u)\mathrm{d}u = \check{Z}(s).$$

Put also $W(s) = \widehat{Z}(s) - \check{Z}(s) = \int_0^s Y(s)$.

For $x \in \mathbb{R}$, we index by "x" the variables conditioned by $X(a) = x$. Z is not Markov, but (Z, X) is Markov, and so (W_x, Y_x) is Markov too, and has a symmetric law, as the subtraction of two independent Markov processes with the same law. The *zero-one law* ensures us again that a.s. W is strictly negative arbitrarily close to 0, and (6) is satisfied for Z. So a.s $a \notin \mathbf{E}_Z^+$. Thanks to lemma 3.1 we can conclude that a.s $\lambda(\mathbf{E}_Z^+) = 0$. We arrive at $\lambda(\mathbf{E}_Z) = 0$ a.s. □

4.2 Itô processes

Proof of theorem 3.2. X is written under the form

$$X(a) = \int_0^a \phi(s)\mathrm{d}B(s) + \int_0^a \psi(s)\mathrm{d}s \tag{7}$$

where $(B, \mathcal{F})$ is a standard brownian motion with its filtration and $\phi, \psi \in \mathcal{M}_2^{loc}$. Denote by $Y(a)$ the first term in (7) and $Z(a)$ the second one. Recall that, by hypothesis, almost surely, Y is null on no interval.

A preliminary remark is that $R^+(X)$ is realised if we can find f continuous non-decreasing on $\mathbb{R}^+$ such that $R^+(X \circ f)$ is realised. Moreover, f can be random. An ideal candidate for this random time change is the function given by the following theorem: (See [9], ch.V.1):

Theorem 4.1. *(Dubins-Schwarz)*
Let M be a local continuous martingale for $\mathcal{F}$. Let $\{\langle M\rangle_a;\ a \in I\}$ be its quadratic variation. Define

$$\tau(a) = \inf\{s;\ \langle M\rangle_s \geq a\}.$$

Then $W = M \circ \tau$ is a standard brownian motion for the filtration $\{\mathcal{F}_{\tau(a)};\ a \in I\}$.

This theorem applies to Y which is a local continuous martingale. So that the time change of Y is continuous, we need that $a \to \langle Y\rangle_a$ is constant on no interval. The process Y being a local continuous martingale, on any deterministic interval where $\langle Y\rangle_a$ is constant, Y has bounded variation and is hence null. Thus the hypotheses imply that τ is continuous.

We apply the time change to Y and keep the same notations.

$$X \circ \tau(a) = W(a) + Z \circ \tau(a).$$

We know that, for any positive a, $\limsup_{v\downarrow 0} \frac{W(a+v)-W(a)}{v} = \infty$. So, if $Z \circ \tau$ is locally lipschitzian, its contribution in the increasing rate is negligible before that of W, and

$$\limsup_{v\downarrow 0} \frac{X \circ \tau(a+v) - X \circ \tau(a)}{v} = \infty.$$

We finally have
$$\mathbb{P}(R_a^-(X)) \leq \mathbb{P}(a \in L_\tau^c),$$
where L_τ is the set of points where τ is locally lipschitzian. (Z is a.s an absolutely continuous function, hence locally lipschitzian everywhere). Using Fubini yields:

$$\int_\Omega \lambda(\{a\,;R_a^-(X)\text{ is realised}\})\mathbb{P}(d\omega) \leq \int_0^\infty \mathbb{P}(a \in L_\tau^c)\mathrm{d}a \leq \int_\Omega \lambda(L_\tau^c)\mathbb{P}(d\omega).verif$$

We need a well known result.

Lemma 4.1 (Riesz-Naguy)**.** *If f is a non-decreasing real function, it is differentiable almost everywhere.*

Hence, τ is locally lipschitzian a.e and a.s $\lambda(L_\tau^c) = 0$, which imply that for all a, $R_a^-(X)$ is almost never realised realised and X takes positive values arbitrarily close to 0.

As $X, -X, \check{X}$ and $-\check{X}$ satisfy the same hypotheses, they also don't realise R_a^- in any a. We can then say that for any a, a.s $M_a = (a, X^*(a))$ is strictly included in the convex polygon $M_{a-s_1}M_{a-u_1}M_{a+s_2}M_{a+u_2}$ for some $u_1, u_2, s_1, s_2 > 0$, and so is not in $\mathcal{E}_X$.

□

4.3 Lévy processes with smooth drift

For the proof of theorem 3.3, X denotes a Lévy process with bounded variation and no drift on I, f is a concave function of class $\mathcal{C}^1$ on I, and $Y = X + f$.

The proof is based on the use of some specific stopping times, defined in the following.

Definition 4.2. *For $\mu > 0$, we set $S_{1,\mu}(Y) = \inf\{a > 0\ ;\ Y(a) > (f'(0)+\mu)a\}$, and $S_{k+1,\mu} = S_{1,\mu} \circ \theta_{S_{k,\mu}} + S_{k,\mu}$, $k \in \mathbb{N}^*$.*

For $u \in \mathbb{Q} \cap I$, we set $S_{k,\mu,u} = S_{k,\mu} \circ \theta_u$. The times $S_{k,\mu,u}$ are called exceeding times.

Those times can are called *exceeding times*, because they correspond to moments where $Y(a+s)$ exceeds the line with slope $f'(a) + \mu$ passing through $(a, Y(a))$, for some a.

Proposition 4.1. *For each $\mu > 0, u \in \mathbb{Q} \cap I$, $\{S_{k,\mu,u};\ k \geq 1\}$ is a sequence of stopping times that goes to ∞.*

Proof. We are going to show it only for $\mu = u = 0$, the general case can be easily deduced.

The stopping time aspect doesn't raise any problem. We define
$$H_{k+1,\mu} = \inf\{a > S_{k,\mu}\ ;\ X(a) - X(S_{k,\mu}) > \mu(a - S_{k,\mu})\}.$$

By concavity,

$$f(S_{k+1,\mu}) - f(S_{k,\mu}) \leq f'(S_{k,\mu})(S_{k+1,\mu} - S_{k,\mu}). \tag{8}$$

We also have, by definition of $S_{k+1,\mu}$,

$$\begin{aligned} X(S_{k+1,\mu}) + f(S_{k+1,\mu}) - X(S_{k,\mu}) - f(S_{k,\mu}) \\ \geq (f'(S_{k,\mu}) + \mu)(S_{k+1,\mu} - S_{k,\mu}). \end{aligned} \tag{9}$$

So, the subtraction (8)-(9) yields,

$$\begin{aligned} X(S_{k+1,\mu}) - X(S_{k,\mu}) \geq (f'(S_{k,\mu}) + \mu)(S_{k+1,\mu} - S_{k,\mu}) - f'(S_{k,\mu})(S_{k+1,\mu} - S_{k,\mu}) \\ \geq \mu(S_{k+1,\mu} - S_{k,\mu}), \end{aligned}$$

and so $S_{k+1,\mu} \geq H_{k+1,\mu}$ by definition of $H_{k+1,\mu}$. Since $\{H_{k+1,\mu} - S_{k,\mu};\, k \geq 1\}$ is a sequence of iid random variables strictly positive, their sum tends a.s to infinity; and so does the sum of the $\{S_{k+1,\mu} - S_{k,\mu};\, k \geq 1\}$, which is what we had to show.

□

We call $\mathbf{J}_X^+$ the set of positive jumping times of X, containing 0 and ∞ by convention.

Proposition 4.2. *We can find a countable set Λ dense in $\mathbb{R}_+^*$ such that a.s, $\{S_{k,\mu,u};\, u \in \mathbb{Q} \cap I, k \in \mathbb{N}, \mu \in \Lambda\} \subset \mathbf{J}_X^+$.*

Proof. We set $T_z = \inf\{a \in I \,;\, X(a) \geq z\}$. Since X is a pure jump process, the Fubini theorem yields

$$\mathbb{E}\lambda(\{z;\, T_z \notin \mathbf{J}_X^+\}) = 0.$$

In particular, there exists a countable set $\Gamma \subset \mathbb{R}_+^*$ such that a.s the T_z, $z \in \Gamma$ are positive jumping times of X (and Y), for $a \in \Gamma$. For $\mu > 0$, we set

$$h(\mu) = (f'(0) + \mu)S_{1,\mu} - f(S_{1,\mu})$$

so that the time where Y exceeds the line with slope $f'(0) + \mu$, X crosses the level $h(\mu)$. The exceeding time $S_{1,\mu}$ also satisfies $S_{1,\mu} = T_{h(\mu)}$. Hence we have

$$h\left(\{\mu \,;\, S_{1,\mu} \notin \mathbf{J}_X^+\}\right) \subset \{z \,;\, T_z \notin \mathbf{J}_X^+\}$$

and so, a.s.,

$$\lambda\left(h\left(\{\mu \,;\, S_{1,\mu} \notin \mathbf{J}_X^+\}\right)\right) = 0.$$

We are going to show that for each $\mu_0 > 0$, $\lambda(\{\mu > \mu_0 \,;\, S_{1,\mu} \notin \mathbf{J}_X^+\}) = 0$ a.s. and then we will have $\lambda(\{\mu \,;\, S_{1,\mu} \notin \mathbf{J}_X^+\}) = 0$ by making μ_0 tend to 0.

Choose $\mu' > \mu > \mu_0$. We have

$$S_{1,\mu'} \geq S_{1,\mu} \text{ and } f(S_{1,\mu'}) - f(S_{1,\mu}) \leq f'(0)(S_{1,\mu'} - S_{1,\mu})$$

by concavity. So,

$$h(\mu') - h(\mu) \geq \mu' S_{1,\mu'} - \mu S_{1,\mu} \geq S_{1,\mu}(\mu' - \mu) \geq S_{1,\mu_0}(\mu' - \mu),$$
$$S_{1,\mu_0}\lambda(\{\mu > \mu_0 \ ; \ S_{1,\mu} \notin \mathbf{J}_X^+\}) \leq \lambda(h(\{\mu > \mu_0 \ ; \ S_{1,\mu} \notin \mathbf{J}_X^+\})) = 0,$$

and, since $S_{1,\mu_0} > 0$, we deduce that, a.s.,

$$\lambda(\{\mu > \mu_0 \ ; \ S_{1,\mu} \notin \mathbf{J}_X^+\}) = 0,$$

which is what we had to show.

Using Fubini, we can then find a countable dense set $\Lambda \subset \mathbb{R}_*^+$ such that a.s, for any $\mu \in \Lambda, u \in \mathbb{Q} \cap I, S_{1,\mu,u} \in \mathbf{J}_X^+$. □

Up to restricting the universe Ω to a smaller set of complete measure, we suppose from now on that, for all $\omega \in \Omega$, for all $\mu \in \Lambda, u \in \mathbb{Q} \cap I$, $S_{1,\mu} \in \mathbf{J}_X^+$.

Proposition 4.3. *Take $\omega \in \Omega$. Let a be an extremal superior time such that $f'(a) < \overline{Y}'(a^-)$. Then we can find $u \in \mathbb{Q} \cap I, k \in \mathbb{N}, \mu \in \Lambda$ such that $a = S_{k,\mu,u}$. In particular a is a positive jumping time.*

Proof. We choose $\mu \in \Lambda$ such that $f'(a) < f'(a) + \mu < \overline{Y}'(a^-)$ and $u \in \mathbb{Q} \cap I$ such that for each $s \in [u, a]$ we have $f'(s) + \mu < \overline{Y}'(a^-)$. It is possible because f' is continuous.

Pick k such that $S_{k,\mu,u} \leq a < S_{k+1,\mu,u}$. If $S_{k,\mu,u} < a$, then, since $a < S_{k+1,\mu,u}$,

$$Y^*(a) < Y(S_{k,\mu,u}) + (a - S_{k,\mu,u})(f'(S_{k,\mu,u}) + \mu).$$

The concavity of $\overline{Y}$ yields

$$\overline{Y}(a) \geq \overline{Y}(S_{k,\mu,u}) + (a - S_{k,\mu,u})\overline{Y}'(a^-) \geq Y(S_{k,\mu,u}) + (a - S_{k,\mu,u})(f'(S_{k,\mu,u)+\mu}).$$

The point a being extremal superior, $\overline{Y}(a) = Y^*(a)$, thus the two last inequalities are contradictory, and we have $S_{k,\mu,u} = a$. □

Proof of theorem 3.3. Let $a \in \mathbf{E}_X^+$ such that $f'(a) \neq \overline{Y}'(a^-)$.
If $f'(a) < \overline{Y}'(a^-)$, we have just seen that $a \in \mathbf{J}_X^+$, thus it is left isolated in $\mathbf{E}_X^+$.

If $f'(a) > \overline{Y}'(a^-)$, applying the same logic to the reversed process $\breve{Y}$, it is clear that a is a positive jumping time for $\breve{Y}$, thus a negative jumping time for Y. Since the set of all jumping times is a countable set of markovian times, by lemma 1.1 we have $\lim_{s\downarrow 0} \frac{Y(b-s)-Y(b)}{s} = f'(b)$ simultaneously for all jumping times b. By hypothesis $f'(a) > \overline{Y}'(a^-)$, so the graph of Y on the left of a is located under the line passing through M_a with slope $\overline{Y}'(a^-)$, and a is left isolated in $\mathbf{E}_Y^+$.

We can apply a similar argument on the right of a.

□

4.4 Lévy processes with no drift

Here, X is a Lévy process with infinite Lévy measure, bounded variation and no drift on an interval I of $\mathbb{R}$. Put

$$T = \inf\{a \in I\ ;\ X^*(a) = \sup_{s\in I} X(s)\}.$$
$$T' = \sup\{a \in I\ ;\ X^*(a) = \sup_{s\in I} X(s)\}.$$

We keep the notation and the intermediate results of the previous proof. In particular, propositions 4.2 and 4.3 are still valid in our case, and since every extremal time is an exceeding time, every extremal time is a jumping time, with probability 1.

Proposition 4.4. *Almost surely,* $T = T'$.

Proof. We set, for $u \in \mathbb{Q} \cap I$,

$$\begin{aligned} m_u^- &= \sup_{s<u} X(s), \\ m_u^+ &= \sup_{s>u} X(s), \\ T_u &= \inf\{a < u\ ;\ X^*(a) = m_u^-\}, \\ T'_u &= \sup\{a > u\ ;\ X^*(a) = m_u^+\}. \end{aligned}$$

The key point is that, if $T \neq T'$, then we can find $u \in \mathbb{Q}\cap]T, T'[$ such that $m_u^- = m_u^+$. Conditionally to $X(u) = a \in \mathbb{R}$, m_u^- and m_u^+ are two independent diffuse variables greater than a, so a.s $m_u^- \neq m_u^+$. Hence $T = T'$. □

We call $\mathbf{J}_X^+$ the set of positive jumping times of X, containing 0 and ∞ by convention. We set $T_z = \inf\{a \in I\ ;\ X(a) \geq z\}$. Since X is a pure jump process, the Fubini theorem yields

$$\mathbb{E}\lambda(\{z;\, T_z \notin \mathbf{J}_X^+\}) = 0.$$

In particular, there exists a countable set $\Gamma \subset \mathbb{R}_+^*$ such that a.s the T_z, $z \in \Gamma$ are positive jumping times of X, for $a \in \Gamma$.

Lemma 4.2. *If T is a positive jumping time, then ν is downwards dissymmetrical, and T is isolated on the left but not on the right.*

Proof. Let us assume that T is a positive jumping time. We saw that it was a stopping time because it is the time of crossing of $z \in \Gamma\cap] \sup_{s<T} X(s), X(T)[$. By definition of T, $\mathbb{P}(R_T^-(X)) = 1 = \mathbb{P}(R^-(X))$, and ν is hence downwards dissymmetrical. Moreover, $\lim\limits_{a\to 0^+} \frac{X(T+a)-X(T)}{a} = 0$ because it is a stopping time, and T is not right isolated. T is left isolated as a positive jump. Considering $\breve{X}$ leads us to the same conclusions on the right of T. □

We have the converse:

Lemma 4.3. *If ν is downwards dissymmetrical, then a.s T is a positive jumping time.*

Proof. Let $V_1 = \inf\{a > 0 \ ; \ X(a) \geq 0\}$ and $V_{n+1} = V_1 \circ \theta_{V_n} + V_n$, it is a sequence of stopping times with iid increments. The measure ν being downwards dissymmetrical, the times V_n are a.s strictly positive. In particular, a.s $\lim_n V_n = \infty$ and the times V_n are discrete in $\mathbb{R}^+$.

Let us also show that $X_{V_1} > 0$. It suffices to notice that if $X_{V_1} = 0$, if we restrict ourselves to an interval of the form $[0, q]$, $q \in \mathbb{Q} \cap]V_1, V_2[$, X reaches twice its maximum on this interval, which almost never happens according to proposition 4.4 . V_1 is then a positive jumping time, and so are the V_n.

It remains to show that $T = V_n$ for a certain n a.s, and that T will then be a.s. a positive jumping time. If T is left isolated in $\{V_n \ ; \ n \in \mathbb{N}\}$, then $T = V_{n+1}$, where $n = \sup\{n \geq 0; V_n < T\}$ since $X_T \geq X_{V_n}$. Otherwise, $T \in \{V_n \ ; \ n \in \mathbb{N}\}$ since this set is discrete. □

We can conclude the demonstration of the theorem by this last lemma:

Lemma 4.4. *We suppose that ν is non-dissymmetrical.*
Then a.s $\overline{X}'(T^-) = \overline{X}'(T) = 0$ and T is isolated neither on its right nor on its left.

Proof. Let A be the event $\overline{X}'(T^-) > 0$. We are going to prove by contradiction that A is negligible.

We suppose here without loss of generality that $I = [0, 1]$ and that X is extended in a stationary way to all the real line. We suppose $\mathbb{P}(A) > 0$. We set $T_2 = \sup\left\{a \in [1, 2] \ ; \ X^*(a) = \sup_{u \in [1,2]} X(u)\right\}$. It is the (a.s unique) time where the restriction of X to $[1, 2]$ reaches its maximum. We call B the sub-event of A:

$$B = \{\omega \in A \ ; \ X(T) < X(T_2) < X(T) + \overline{X}'(T^-)(T_2 - T)\}.$$

The event B occurs when the three following events are realised.

(i) X reaches its maximum in T on $[0, 1]$,

(ii) X reaches its maximum on $[0, 2]$ on $T_2 \neq T$,

(iii) M_T is in $\mathcal{E}_{X_{|[0,2]}}$.

Let $u \in]0, 1[$, $M > m > 0$, $c > 0$, and let Z be the zone of $\mathbb{R}^2$ defined by

$$Z = \{(a, x) \in \mathbb{R}^2 \ ; \ 1 < a < u, M < x < m + c(a - u)\}.$$

We suppose that m and M have been chosen such that $\lambda^2(Z) > 0$, where λ^2 is the 2-dimensional Lebesgue measure. Conditionally to the event

$$C = (A, T < u, m < X(T) < M, X'(T^-) > c, X(T_2) > X(T)),$$

the Lévy measure being infinite, M_{T_2} has a diffuse law whose support contains Z. We have

$$\mathbb{P}(B\ ;\ C) \geq \mathbb{P}(M_{T_2} \in Z\ ;\ C) > 0,$$

and so

$$\mathbb{P}(B) > 0.$$

However, if A and B are simultaneously realised, then T is an extremal superior time for X on $[0,2]$, while being strictly inferior to the time T_2 where X reaches its maximum on $[0,2]$. The time T is then a positive jumping time of X, which almost never happens according to lemma 4.2 since we supposed ν non-dissymmetrical, and we arrive at a contradiction.

We hence proved $\overline{X}'(T^-) = 0$, and by reflexivity we also have $\overline{X}'(T) = 0$. Since T is the only point where X reaches its maximum, it implies that T is isolated neither on its right nor on its left in $\mathbf{E}_X^+$. □

Acknowledgements

I wish to thank Youri Davydov, my PhD advisor, at the origin of this paper, for his guidance in this work.

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